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Continua whose hyperspace of nonblockers of $\mathcal{F}_1(X)$ is a continuum

Javier Camargo [a], Félix Capulín [b], Enrique Castañeda-Alvarado [b], David Maya [b,*]

[a] Escuela de Matemáticas, Facultad de Ciencias, Universidad Industrial de Santander, Ciudad Universitaria, Carrera 27 Calle 9, Bucaramanga, Santander, A. A. 678, Colombia
[b] Universidad Autónoma del Estado de México, Facultad de Ciencias, Instituto Literiario 100, Col. Centro, Toluca, CP 50000, México

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ABSTRACT

A *continuum* is a compact connected metric space. A non-empty closed subset B of a continuum X *does not block* $x \in X \setminus B$ provided that the union of all subcontinua of X containing x and contained in $X \setminus B$ is a dense subset of X. The collection of all non-empty closed subsets B of X such that B does not block each element of $X \setminus B$ is denoted by $\mathcal{NB}(\mathcal{F}_1(X))$. In this paper, we find conditions under which $\mathcal{NB}(\mathcal{F}_1(X))$ and the hyperspace of non-weak cut sets $\mathcal{NWC}(X)$ coincide and we exhibit a dendroid X for which $\mathcal{NWC}(X)$ is a non-empty proper subset of $\mathcal{NB}(\mathcal{F}_1(X))$. Also, we present geometric models for $\mathcal{NB}(\mathcal{F}_1(X))$; particularly, some of them give examples for a question posed by Escobedo, López and Villanueva in 2012. Finally, we prove that there exists a family of continua X such that the collection of hyperspaces $\mathcal{NB}(\mathcal{F}_1(X))$ is an uncountable incomparable family of continua.

1. Introduction

A *continuum* is a compact connected metric space. For a continuum X, let 2^X be the hyperspace of all non-empty closed subsets of X and let $\mathcal{F}_1(X)$ be the hyperspace of all one-point subsets of X.

Given a continuum X, $A \in 2^X$ and $x \in X \setminus A$, we say that A *does not block* x if the union of all subcontinua of X containing x and contained in $X \setminus A$ is a dense subset of X. The collection of all elements B of 2^X such that B does not block each element of $X \setminus B$ is denoted by $\mathcal{NB}(\mathcal{F}_1(X))$.

The notion of blocker and nonblocker in hyperspaces have been studied recently by many authors (see [2], [3], [4], [5], [6], [8], [10] and [14]). In [8], the notion of blocker is introduced and general properties using different kinds of continua are presented. In [5], the set $\mathcal{NB}(\mathcal{F}_1(X))$ is used to characterize classes

* Corresponding author.
E-mail addresses: jcamargo@saber.uis.edu.co (J. Camargo), fcapulin@gmail.com (F. Capulín), eca@uaemex.mx (E. Castañeda-Alvarado), dmayae@outlook.com, dmayae@uaemex.mx (D. Maya).

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of continua; for instance, it is proved that if X is a locally connected continuum, then $\mathcal{NB}(\mathcal{F}_1(X))$ is a continuum if and only if X is a simple closed curve. So, the following question arises:

Question 1.1. *[5, Question 4.5, p. 3617] For which nonlocally connected continua X is $\mathcal{NB}(\mathcal{F}_1(X))$ a continuum?*

The circle of pseudo-arcs is an example for Question 1.1 whose hyperspace $\mathcal{NB}(\mathcal{F}_1(X))$ is a simple closed curve (see [5, Corollary 5.4, p. 3618] and see [15] for the definition of the circle of pseudo-arcs).

On the other hand, the hyperspace of all non-weak cut sets of a continuum X, denoted by $\mathcal{NWC}(X)$, is contained in $\mathcal{NB}(\mathcal{F}_1(X))$ (see [6, Theorem 3.2, p. 99]) and if X is indecomposable, then $\mathcal{NWC}(X) = \emptyset$. So, the inclusion $\mathcal{NWC}(X) \subseteq \mathcal{NB}(\mathcal{F}_1(X))$ is proper when X is an indecomposable continuum such that $\mathcal{NB}(\mathcal{F}_1(X))$ is a non-empty set (see [6, Remark 3.3, p. 99]).

Our main interests are the following: first, to show that each one of the following spaces: a finite set, the closure of a convergent sequence, the hyperspace 2^Y for each continuum Y, an arc, a 2-cell and Hilbert cube, is homeomorphic to the hyperspace $\mathcal{NB}(\mathcal{F}_1(X))$ for some continuum X; second, to prove that there exists an uncountable family of continua such that each one of them is an example for Question 1.1; and third, to show that the inclusion $\mathcal{NWC}(X) \subseteq \mathcal{NB}(\mathcal{F}_1(X))$ can be proper even when $\mathcal{NWC}(X)$ is non-empty and X is a dendroid.

2. Preliminaries

The set of all positive integers is denoted by the symbol $\mathbb{N}$. The word *mapping* stands for a continuous function between topological spaces. Given a topological space X and a subset A of X, the symbols $\operatorname{Int} A$ and $\operatorname{Cl} A$ represent the interior and the closure of A in X, respectively. A *Cantor set* is a space homeomorphic to the countable Cartesian product $\{0,1\}^{\mathbb{N}}$ of the discrete space $\{0,1\}$. An *arc* is any space homeomorphic to the unit interval $[0,1]$. A continuum homeomorphic to $[0,1]^2$ will be called 2-*cell*. A *Hilbert cube* is a space homeomorphic to the countable Cartesian product $[0,1]^{\mathbb{N}}$.

A continuum is *decomposable* provided that it is the union of two of its proper subcontinua. A continuum that is not decomposable is said to be *indecomposable*.

Given subsets $U_1, U_2, \ldots, U_l$ of a continuum X, we define

$$\langle U_1, U_2, \ldots, U_l \rangle = \left\{ A \in 2^X : A \subseteq \bigcup_{i=1}^{l} U_i \text{ and } A \cap U_i \neq \emptyset \text{ for each } i \right\}.$$

The *Vietoris topology* is the topology on 2^X generated by the base consisting of all subsets of the form $\langle U_1, U_2, \ldots, U_l \rangle$, where $U_1, U_2, \ldots, U_l$ are open subsets of the continuum X (see [11, Proposition 2.1, p. 155]).

A connected element of 2^X is called a *subcontinuum* of the continuum X. The hyperspace of all subcontinua of a continuum X is denoted by $\mathcal{C}(X)$. Observe that if X is a continuum, then $\mathcal{F}_1(X) \subseteq \mathcal{C}(X) \subseteq 2^X$. Thus, the hyperspaces $\mathcal{C}(X)$ and $\mathcal{F}_1(X)$ will be considered as subspaces of 2^X.

For a continuum X and $A, B \in 2^X$ such that $A \subseteq B$, an *order arc from A to B* is a mapping $\alpha : [0,1] \to 2^X$ such that $\alpha(0) = A$, $\alpha(1) = B$ and if $0 \leq s < t \leq 1$, then $\alpha(s) \subsetneq \alpha(t)$. When $\alpha([0,1]) \subseteq \mathcal{C}(X)$, we say that it is *an order arc in $\mathcal{C}(X)$*.

The next result will be used throughout this paper without mentioning explicitly.

Theorem 2.1. *[9, Theorem 14.6, p. 112] Let X be a continuum and let $A, B \in \mathcal{C}(X)$ be such that $A \subsetneq B$. Then there exists an order arc in $\mathcal{C}(X)$ from A to B.*

A set Y of a continuum X is *continuumwise connected* if each two points in Y belong to a subcontinuum of X contained in Y.

For a continuum X, an element $A \in 2^X \setminus \{X\}$:

- is a *non-weak cut set* of X provided $X \setminus A$ is continuumwise connected, and
- *does not block* $B \in 2^X$ if there exists a mapping $\alpha : [0,1] \to 2^X$ such that $\alpha(0) = B$, $\alpha(1) = X$ and $\alpha(t) \cap A = \emptyset$ for each $t \in [0,1)$.

The following result will be used throughout the current paper.

Proposition 2.2. *[5, Proposition 2.2, p. 3615] For a continuum X, $A \in 2^X$ and $x \in X \setminus A$, the following statements are equivalent:*

(a) *A does not block $\{x\}$;*
(b) *there exists an order arc $\alpha : [0,1] \to \mathcal{C}(X)$ from $\{x\}$ to X such that $A \cap \alpha(t) = \emptyset$ for each $t \in [0,1)$;*
(c) *there exists a sequence $(C_n)_{n\in\mathbb{N}}$ in $\mathcal{C}(X)$ such that $x \in C_n \subseteq C_{n+1} \subseteq X \setminus A$ for each $n \in \mathbb{N}$, and $\bigcup\{C_n : n \in \mathbb{N}\}$ is a dense subset of X;*
(d) *The set $\bigcup\{C \in \mathcal{C}(X) : x \in C \subseteq X \setminus A\}$ is dense in X.*

We use the following notation introduced in [6].

Notation 2.3. *Given a continuum X.*

$$\begin{aligned}\mathcal{NWC}(X) =& \{A \in 2^X : \operatorname{Int} A = \emptyset \text{ and } A \text{ is non-weak cut set of } X\},\\ \mathcal{NB}(\mathcal{F}_1(X)) =& \{B \in 2^X : B \text{ does not block } \{x\} \text{ for each } x \in X \setminus B\}.\end{aligned}$$

For any continuum X, for any point $p \in X$ and for any proper subset B of X containing p, let

$$\kappa(p) = \bigcup\{A \in \mathcal{C}(X) : A \neq X, p \in A\}$$

and let

$$\kappa(p, B) = \bigcup\{A \in \mathcal{C}(X) : p \in A \subsetneq B\}.$$

The set $\kappa(p)$ is called the *composant of* p in X. Both $\kappa(p, B)$ and $\kappa(p)$ are continuumwise connected subsets of X.

Theorem 2.4. *[6, Theorem 3.1, p. 99] Let X be a continuum and let $A \in 2^X$ be such that $\operatorname{Int} A = \emptyset$. Each one of the following statement holds.*

(a) *$A \in \mathcal{NWC}(X)$ if and only if $\kappa(x, X \setminus A) = X \setminus A$ for each $x \in X \setminus A$.*
(b) *$A \in \mathcal{NB}(\mathcal{F}_1(X))$ if and only if $\kappa(x, X \setminus A)$ is a dense subset of X for each $x \in X \setminus A$.*

The proof of the next result follows from combining both points of Theorem 2.4.

Theorem 2.5. *Let X be a continuum. Then $\mathcal{NWC}(X) = \mathcal{NB}(\mathcal{F}_1(X))$ if and only if each open subset U of X such that $\overline{\kappa(p, U)} = X$ for each $p \in U$ is continuumwise connected.*

3. Hyperspace of non-cut sets

The aim of this section is to give conditions on a continuum X that imply the equality between $\mathcal{NWC}(X)$ and $\mathcal{NB}(\mathcal{F}_1(X))$ and to show that the inclusion in the following result can be proper being both sets non-empty in contrast to [6, Remark 3.3, p. 99].

Theorem 3.1. *[6, Theorem 3.2, p. 99] For each continuum X,*

$$\mathcal{NWC}(X) \subseteq \mathcal{NB}(\mathcal{F}_1(X)).$$

Theorem 3.2. *Let X be a continuum. If there exists $p \in X$ with the property that $p \in K$ whenever K is a continuumwise connected dense subset of X, then $\mathcal{NWC}(X) = \mathcal{NB}(\mathcal{F}_1(X))$.*

Proof. In light of Theorem 3.1, it only remains to prove that $\mathcal{NB}(\mathcal{F}_1(X))$ is contained in $\mathcal{NWC}(X)$. Let $A \in \mathcal{NB}(\mathcal{F}_1(X))$. By (b) of Theorem 2.4, each $\kappa(x, X \setminus A)$ is a continuumwise connected dense subset of X. Then $p \in \kappa(x, X \setminus A)$ for each $x \in X \setminus A$ and $\operatorname{Int} A = \emptyset$. Thus, $\kappa(x, X \setminus A) = X \setminus A$ for each $x \in X \setminus A$. From (a) of Theorem 2.4, it follows that $A \in \mathcal{NWC}(X)$. □

A point q in a continuum X is called a *strong center* if there exist non-empty open subsets U and V of X such that each subcontinuum of X intersecting both sets U and V contains q.

Corollary 3.3. *Let X be a continuum. If X has a strong center, then $\mathcal{NWC}(X) = \mathcal{NB}(\mathcal{F}_1(X))$.*

Proof. Assume that q is a strong center point of X. Let U and V be open subsets of X such that if $L \in \mathcal{C}(X)$ intersects both U and V, then $q \in L$. Observe that for each continuumwise connected dense subset K of X, there exists $L \in \mathcal{C}(X)$ such that $L \subseteq K$, $L \cap U \neq \emptyset$ and $L \cap V \neq \emptyset$ and so, $q \in K$. The corollary follows from Theorem 3.2. □

A point q in a continuum X is called a *cut point* provided that $X \setminus \{q\}$ is not connected.

The result below is an immediate consequence of the fact that each cut point of a continuum X is a strong center of X and Corollary 3.3.

Corollary 3.4. *Let X be a continuum. If X has a cut point, then $\mathcal{NWC}(X) = \mathcal{NB}(\mathcal{F}_1(X))$.*

A *dendroid* is an arcwise connected and hereditarily unicoherent continuum (*hereditarly unicoherent* means that the intersection of two subcontinua is connected). A dendroid having a unique ramification point (i.e., with only one point which is the common part of three otherwise disjoint arcs) is called a *fan*.

Corollary 3.5. *If X is a planar dendroid, then $\mathcal{NWC}(X) = \mathcal{NB}(\mathcal{F}_1(X))$.*

Proof. From [7, Corollary 4, p. 180], X has a strong center. So, the result follows from Corollary 3.3. □

Corollary 3.6. *If X is a fan, then $\mathcal{NWC}(X) = \mathcal{NB}(\mathcal{F}_1(X))$.*

Proof. The vertex v of X is a strong center of X. Invoke Corollary 3.3 to prove that $\mathcal{NWC}(X)$ and $\mathcal{NB}(\mathcal{F}_1(X))$ are equal. □

On other hand, in [6, Remark 3.3, p. 99] it is shown that the inclusion of Theorem 3.1 can be proper in the case when X is an indecomposable continuum such that $\mathcal{NB}(\mathcal{F}_1(X)) \neq \emptyset$ because $\mathcal{NWC}(X) = \emptyset$. Naturally, the question below arises:

Question 3.7. *Is there a continuum X such that $\mathcal{NWC}(X)$ is a non-empty proper subset of $\mathcal{NB}(\mathcal{F}_1(X))$?*

This question will be answered in the positive form by exhibiting a dendroid X having the required property in the following example.

Example 3.8. Let us denote by C the Cantor set. Given $k \in \mathbb{N}$, define the mappings $\rho_k : C \to \{0,1\}^k$ and $\varphi_k : C \to C$ by $\rho_k((c_n)_{n\in\mathbb{N}}) = (c_1, \cdots, c_k)$ and $\varphi_k((c_n)_{n\in\mathbb{N}}) = (c_n)_{n\geq k}$, respectively. Let $\mathfrak{D}$ be the partition of $C \times [0,1]$ whose unique non-degenerate members are the set $C \times \{1\}$ and all finite sets of the form

$$\varphi_k^{-1}(c) \times \{t\}$$

where $k \in \mathbb{N}$, $t \in [1 - \frac{1}{k+1}, 1 - \frac{1}{k+2})$ and $c \in C$. Observe that $\mathfrak{D}$ is an upper semicontinuous decomposition. So, in light of [13, Theorem 3.9, p. 40], the decomposition space, which is denoted by X, is a compact metrizable space. Notice that X is connected. Then X is a continuum. Moreover, X is a dendroid as well. For sake of simplicity, the quotient mapping from $C \times [0,1]$ onto X will be denoted by q and v will be the unique point in $q(C \times \{1\})$ (see Fig. 1.)

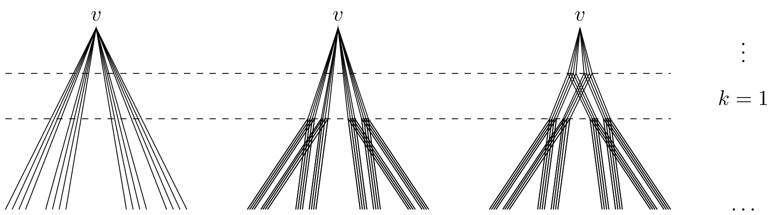

Fig. 1. Construction of dendroid X.

In order to see that $\mathcal{NWC}(X)$ is non-empty, let $z \in C$ be arbitrary. If $(a,t) \in (C \times [0,1]) \setminus \{(z,0)\}$, then the connected subset $q(\{(a,s) : s \in [t,1]\})$ of X contains $q((a,t))$ and v but omits $q((z,0))$. From this, the equality$\kappa((b,r), X \setminus \{q((z,0))\}) = X \setminus \{q((z,0))\}$ holds for each $(b,r) \in (C \times [0,1]) \setminus \{(z,0)\}$. By (a) of Theorem 2.4, we conclude that $q((z,0)) \in \mathcal{NWC}(X)$.

Now, we will argue that $\{v\} \in \mathcal{NB}(\mathcal{F}_1(X)) \setminus \mathcal{NWC}(X)$. Let $(x,t) \in C \times [0,1)$ be arbitrary and let U be a non-empty open subset of X. Since the family $\{\rho_k^{-1}(\rho_k(c)) : (c,k) \in C \times \mathbb{N}\}$ is a basis for C, there exist $m \in \mathbb{N}$, $w \in C$ and $s \in [0,1)$ such that $\rho_m^{-1}(\rho_m(w)) \times \{s\} \subseteq q^{-1}(U)$. Let y be the unique point in the set $\rho_m^{-1}(\rho_m(w)) \cap \varphi_{m+1}^{-1}(\varphi_{m+1}(x))$. If $k \in \mathbb{N}$ satisfies that $k \geq m$ and $t, s \leq 1 - \frac{1}{k+1}$, then $q((y,s)) \in U$ and the connected set $q(\{x\} \times [t, 1 - \frac{1}{k+1}]) \cup q(\{y\} \times [s, 1 - \frac{1}{k+1}])$ contains $q((x,t))$ and $q((y,s))$ and is contained in $X \setminus \{v\}$. This proves that $\kappa((x,t), X \setminus \{v\})$ is dense in X. By (b) of Theorem 2.4, we conclude that $\{v\} \in \mathcal{NB}(\mathcal{F}_1(X))$. Now, let $a, b \in C$ be such that $b \notin \varphi_k^{-1}(\varphi_k(a))$ for each $k \in \mathbb{N}$. From this condition, it follows that $q(b,t) \neq q(a,t)$ for any $t \in [0,1]$. Thus, v must belong to the unique arc between $q(a,0)$ and $q(b,0)$. So, since X is a dendroid, each subcontinuum B of X containing $q((a,0))$ and $q((b,0))$ must contain the arc between $q((a,0))$ and $q((b,0))$ and this implies that $v \in B$. Hence, $q((b,0)) \notin \kappa(q((a,0)), X \setminus \{v\})$. Invoke (a) of Theorem 2.4 to prove that $\{v\} \notin \mathcal{NWC}(X)$.

4. $\mathcal{NB}(\mathcal{F}_1(X))$ is a continuum

In this section, we present some geometric models for the hyperspace$\mathcal{NB}(\mathcal{F}_1(X))$. Particularly, we show some continua X, for which $\mathcal{NB}(\mathcal{F}_1(X))$ is a continuum giving a partial answer to Question 1.1. In order to give clear arguments to our examples, new concepts will be introduced and some results about them will be presented.

The next result contains some basic facts about nonblockers.

Proposition 4.1. *[5, Remark 2.1, p. 3615] Let X be a continuum. If $A \in \mathcal{NB}(\mathcal{F}_1(X))$, then each one of the following statements holds:*

(a) $\operatorname{Int} A = \emptyset$.
(b) $X \setminus H$ *is connected for every* $H \subseteq A$.

Let Z be a subcontinuum of a continuum X. Then Z is called *quasi-terminal* provided that there exists a point $z \in Z$ such that for each $Y \in \mathcal{C}(X)$ that intersects $\kappa(z, Z)$, either $Y \subseteq Z$ or $Z \subseteq Y$. If the condition $Y \cap Z \neq \emptyset$ implies that either $Y \subseteq Z$ or $Z \subseteq Y$ for each $Y \in \mathcal{C}(X)$, then Z is called *terminal*. Every terminal continuum is quasi-terminal. Furthermore, if $X = M \cup L$, $|M \cap L| = 1$, M is an indecomposable continuum and L is a non-degenerate continuum, then X is a continuum and M is quasi-terminal and non-terminal subcontinuum of X.

Proposition 4.2. *Let X be a continuum and let Y be a proper subcontinuum of X. If Y is decomposable, then Y is terminal if and only if Y is quasi-terminal.*

Proof. Assume that Y is quasi-terminal. Let $y \in Y$ be such that if $Z \in \mathcal{C}(X)$ is such that $Z \cap \kappa(y, Y) \neq \emptyset$, then either $Y \subseteq Z$ or $Z \cap Y$ and let A and B be proper subcontinua whose union is Y. In order to prove that Y is terminal, let $K \in \mathcal{C}(X)$ be such that $K \cap Y \neq \emptyset$. We shall show that either $K \subseteq Y$ or $Y \subseteq K$.

First, observe that $Y \setminus A$ and $Y \setminus B$ are non-empty open subsets of Y such that $Y \setminus A \subseteq B$ and $Y \setminus B \subseteq A$. From this and [13, 5.20, (a), p. 83], it follows that $A \cap \kappa(y, Y) \neq \emptyset$ and $B \cap \kappa(y, Y) \neq \emptyset$. Now, since $Y \cap K \neq \emptyset$, we may assume that $K \cap A \neq \emptyset$. Then $K \cup A \in \mathcal{C}(X)$ fulfils $(K \cup A) \cap \kappa(y, Y) \neq \emptyset$. Our assumption implies that either $K \cup A \subseteq Y$ or $Y \subseteq K \cup A$. The inclusion $K \cup A \subseteq Y$ guarantees that $K \subseteq Y$. Next, assume that $Y \subseteq K \cup A$. Notice that if $b \in B \setminus A$, then $b \in Y \setminus A$ and so b must belong K. Hence, $K \cap B \neq \emptyset$. Finally, since either $A \subseteq \kappa(y, Y)$ or $B \subseteq \kappa(y, Y)$, we have that $K \cap \kappa(y, Y) \neq \emptyset$. From this, it follows that either $Y \subseteq K$ or $K \subseteq Y$. □

Lemma 4.3. *Let X be a continuum. If Z is a quasi-terminal subcontinuum of X and $A \in \mathcal{NB}(\mathcal{F}_1(X))$, then either $Z \subseteq A$ or $Z \cap A = \emptyset$.*

Proof. Suppose that $Z \setminus A \neq \emptyset$. Since each set $\kappa(y, Z)$ is dense in Z (see [13, 5.20, (a), p. 83]), by our assumption on Z, we may choose $z \in Z \setminus A$ such that for each $Y \in \mathcal{C}(X)$ fulfilling $Y \cap \kappa(z, Z) \neq \emptyset$, either $Y \subseteq Z$ or $Z \subseteq Y$. On other hand, by Proposition 2.2, there exists an order arc $\alpha : [0, 1] \to \mathcal{C}(X)$ such that $\alpha(0) = \{z\}$, $\alpha(1) = X$ and $\alpha(t) \cap A = \emptyset$ for each $t < 1$. The continuity of α and the fact that $\alpha(1) = X \in \langle X, X \setminus Z\rangle$ imply that there exists $t_0 < 1$ such that $\alpha(t_0) \in \langle X, X \setminus Z\rangle$. Thus, $\alpha(t_0)$ is a proper subcontinuum of X that intersects both sets $\kappa(z, Z)$ and $X \setminus Z$. Then, Z must be contained in $\alpha(t_0)$. This conclusion together with the condition $\alpha(t_0) \cap A = \emptyset$ guarantee that $Z \cap A = \emptyset$. □

A subset V of a continuum X is called *quasi-terminal continuum-wise connected* if for each $x, y \in V$, there exist quasi-terminal subcontinua $C_1, \ldots, C_k$ of X such that $x \in C_1$, $y \in C_k$, $\bigcup_{i=1}^{k} C_i \subseteq V$ and $C_j \cap C_{j+1} \neq \emptyset$ for each $j \in \{1, \ldots, k-1\}$.

Lemma 4.4. *Let X be a continuum. If V is a quasi-terminal continuum-wise connected subset of X and $A \in \mathcal{NB}(\mathcal{F}_1(X))$, then either $A \cap V = \emptyset$ or $V \subseteq A$.*

Proof. Suppose that $A \cap V \neq \emptyset$. Fix $x \in A \cap V$. Let us prove that $V \subseteq A$. Let $y \in V$ be arbitrary. From our assumption on V, it follows that there exist quasi-terminal subcontinua $C_1, \ldots, C_k$ of X satisfying that $x \in C_1$, $y \in C_k$, $\bigcup_{i=1}^{k} C_i \subseteq V$ and $C_j \cap C_{j+1} \neq \emptyset$ for each $j \in \{1, \ldots, k-1\}$. In light of Lemma 4.3, since

$x \in A \cap C_1$, we have that $C_1 \subseteq A$. Hence, $A \cap C_2 \neq \emptyset$. Applying again Lemma 4.3, we get that $C_2 \subseteq A$. Similarly, each inclusion $C_j \subseteq A$ holds. Then $y \in C_k \subseteq A$. Thus, V is contained in A. □

A proper closed subset B of a continuum X is called *co-quasi-terminal continuum-wise* provided that each component of $X \setminus B$ is a quasi-terminal continuum-wise connected open subset of X.

Proposition 4.5. *Let X be a continuum and let B be a proper closed subset of X. If B is a co-quasi-terminal continuum-wise connected subset of X, then $\mathcal{NB}(\mathcal{F}_1(X)) \subseteq \langle B \rangle$.*

Proof. Let $A \in \mathcal{NB}(\mathcal{F}_1(X))$ be arbitrary. Assume that A is not contained in B. If V is a component of $X \setminus B$ such that $A \cap V \neq \emptyset$, then by Lemma 4.4 the inclusion $V \subseteq A$ holds. Hence, $\operatorname{Int} A$ is non-empty which contradicts Proposition 4.1. □

Given a continuum X and a subset E of X, we will write $D(X, E) = 2$ provided that each subset of E having exactly two points disconnects X (compare [13, p. 148]).

Proposition 4.6. *Let X be a continuum and let E be a subset of X. If $D(X, E) = 2$, then $\mathcal{NB}(\mathcal{F}_1(X)) \cap \langle E \rangle \subseteq \mathcal{F}_1(E)$.*

Proof. Let $A \in \mathcal{NB}(\mathcal{F}_1(X)) \cap \langle E \rangle$ be arbitrary. In light of Proposition 4.1, each subset F of A satisfies $X \setminus F$ is connected. By our assumption on E, A must be a singleton. □

Proposition 4.7. *Let X be a continuum and let B be a proper closed subset of X. If B is a co-quasi-terminal continuum-wise connected subset of X, $D(X, B) = 2$ and $\mathcal{F}_1(B) \subseteq \mathcal{NWC}(X)$, then $\mathcal{NB}(\mathcal{F}_1(X)) = \mathcal{F}_1(B)$.*

Proof. The inclusion $\mathcal{NB}(\mathcal{F}_1(X)) \subseteq \langle B \rangle$ is guaranteed by Proposition 4.5. Apply Proposition 4.6 to obtain that $\mathcal{NB}(\mathcal{F}_1(X)) \subseteq \mathcal{F}_1(B)$. Finally, from our assumption $\mathcal{F}_1(B) \subseteq \mathcal{NWC}(X)$ and Theorem 3.1, it follows that$\mathcal{NB}(\mathcal{F}_1(X)) = \mathcal{F}_1(B)$. □

An arc I properly contained in a continuum X *has outlet end points* provided that each $Z \in \mathcal{C}(X)$ satisfying $Z \cap I \neq \emptyset$ and $Z \setminus I \neq \emptyset$ contains at least one end point of I.

Proposition 4.8. *Let X be a continuum. If I is an arc having outlet end points contained in X, then $\mathcal{NB}(\mathcal{F}_1(X)) \cap \langle I \rangle \subseteq \mathcal{C}(I)$.*

Proof. Fix a homeomorphism $h : [0, 1] \to I$. Let $A \in \mathcal{NB}(\mathcal{F}_1(X)) \cap \langle I \rangle$. Set $r = \sup h^{-1}(A)$ and $t = \inf h^{-1}(A)$. Suppose that $h^{-1}(A)$ is a proper subset of $[t, r]$. Let $s \in [t, r]$ be such that $h(s) \notin A$. From (b) of Theorem 2.4, it follows that $\kappa(h(s), X \setminus A) \cap (X \setminus I) \neq \emptyset$. This implies that there exists $Z \in \mathcal{C}(X)$ such that $h(s) \in Z \subseteq X \setminus A$ and $Z \setminus I \neq \emptyset$. Now, let $\alpha : [0, 1] \to \mathcal{C}(X)$ be an order arc such that $\alpha(0) = \{h(s)\}$ and $\alpha(1) = Z$. Set $l = \inf \alpha^{-1}(\langle X, X \setminus I \rangle)$. We have that $h(s) \in \alpha(l) \subseteq I$ and $\alpha(l) \cap \{h(0), h(1)\} \neq \emptyset$. Then either $h([0, s]) \subseteq \alpha(l)$ or $h([s, 1]) \subseteq \alpha(l)$. Hence, $\alpha(l) \cap \{h(t), h(r)\}$ is non-empty. Then $Z \cap A$ must be non-empty. A contradiction.

In conclusion, $A = h([t, r]) \in \mathcal{C}(I)$. □

The purpose of introducing the following notation is to describe all continua in our examples.

For $r \in \mathbb{R}$, define the homeomorphisms $\lambda_r, \varphi_r, \rho_r, \sigma_r : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\lambda_r(a, b) = (ra, rb),$$

$$\varphi_r(a, b) = (r + a, r + b),$$

$$\rho_r(a,b) = (r(2-a), b) \text{ and}$$
$$\sigma_r(a,b) = (r(1+a), b).$$

The points of the plane $\mathbb{R}^2$ of the form (x,x) will be denoted by $\overline{x}$.

Let P be an indecomposable planar continuum such that $P \subseteq [0,1]^2$ and $P \cap (([0,1] \times \{0,1\}) \cup (\{0,1\} \times [0,1])) = \{\overline{0}, \overline{1}\}$. Now, for each $n \in \mathbb{N}$, set $W_n = \varphi_{2^{-n}} \circ \lambda_{2^{-n}}(P)$, $Q_n = \sigma_{2^{-n}}(P)$ if n is odd and $Q_n = \rho_{2^{-n}}(P)$ if n is even. Define

$$W = \{\overline{0}\} \cup \bigcup_{n\in\mathbb{N}} W_n = \mathrm{Cl}_{\mathbb{R}^2}\left(\bigcup_{n\in\mathbb{N}} W_n\right),$$
$$W^* = W \cup \lambda_{-1}(W),\ W^\dagger = W \cup \varphi_2 \circ \lambda_{-1}(W)$$

and

$$Q = (\{0\} \times [0,1]) \cup \bigcup_{n\in\mathbb{N}} Q_n = \mathrm{Cl}_{\mathbb{R}^2}\left(\bigcup_{n\in\mathbb{N}} Q_n\right).$$

See Fig. 2.

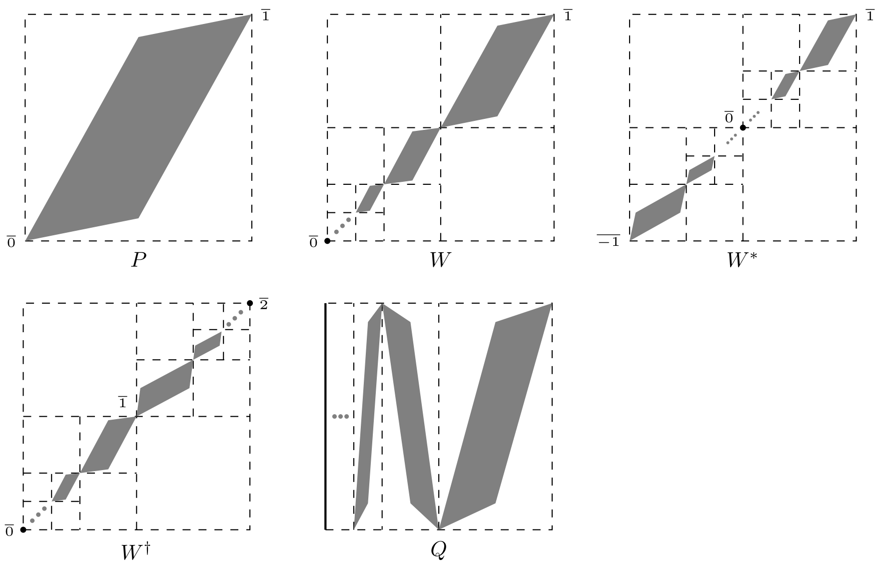

Fig. 2. Models for continua P, W, W^*, $W^\dagger$ and Q.

Example 4.9. For each $k \in \mathbb{N}$, there exists a continuum X such that $\mathcal{NB}(\mathcal{F}_1(X))$ consists of exactly k elements.

First, let us show that $\mathcal{NB}(\mathcal{F}_1(W)) = \{\{\overline{0}\}\}$. Since $W \setminus \{\{\overline{0}\}\}$ is a quasi-terminal continuum-wise connected open subset of W, Proposition 4.5 guarantees that $\mathcal{NB}(\mathcal{F}_1(X))$ is a subset of $\{\{\overline{0}\}\}$. On the other hand, observe that $\kappa(x, W \setminus \{\{\overline{0}\}\}) = W \setminus \{\{\overline{0}\}\}$ for each $x \in W \setminus \{\{\overline{0}\}\}$. This implies that $\{\overline{0}\} \in \mathcal{NWC}(X)$. By Theorem 3.1, we have that $\{\overline{0}\} \in \mathcal{NB}(\mathcal{F}_1(X))$. In conclusion, $\mathcal{NB}(\mathcal{F}_1(W))$ has exactly one element.

Now, let $k \geq 2$. Set $Y = \bigcup_{j=0}^{k-1} \varphi_{2j}(W^*)$. Let X be the quotient space $Y/\{\overline{-1}, \overline{2k-1}\}$ obtained from Y by shrinking the closed set $\{\overline{-1}, \overline{2k-1}\}$ to a point. Then X is a continuum (see [13, 3.14, p. 41]). Denote the natural quotient mapping from Y onto X by q. Observe that the closed subset $B = q(\{\overline{2j} : j \in \{0,\ldots,k-1\}\})$ of X is co-quasi-terminal continuum-wise connected such that $D(X,B) = 2$ and

$\mathcal{F}_1(B) \subseteq \mathcal{NWC}(X)$. In light of Proposition 4.7, $\mathcal{NB}(\mathcal{F}_1(X)) = \mathcal{F}_1(B)$. In conclusion, $\mathcal{NB}(\mathcal{F}_1(X))$ has exactly k elements.

Example 4.10. There exists a continuum X such that $\mathcal{NB}(\mathcal{F}_1(X))$ is homeomorphic to the closure of a convergent sequence.

Set $Y = \lambda_{-1}(W) \cup \bigcup_{n\in\mathbb{N}} \varphi_{3\cdot 2^{-n}} \circ \lambda_{2^{-(n+1)}}(W^*)$. Let X be the quotient space $Y/\{\overline{1}, \overline{-1}\}$ obtained from Y by identifying the points $\overline{1}$ and $\overline{-1}$. The space X is a continuum (see [13, 3.14, p. 41]). Let q be the natural quotient mapping from Y onto X and set $B = q(\{\overline{0}\} \cup \{\overline{3\cdot 2^{-(n+1)}} : n \in \mathbb{N}\})$. Then B is the closure of a convergent sequence in X. On the other hand, B is a co-quasi-terminal continuum-wise connected subset of X that satisfies that $D(X, B) = 2$ and $\mathcal{F}_1(X) \subseteq \mathcal{NWC}(X)$. So, from Proposition 4.7, it follows that $\mathcal{NB}(\mathcal{F}_1(X)) = \mathcal{F}_1(B)$.

Example 4.11. There exists a continuum X containing an arc I such that $\mathcal{NB}(\mathcal{F}_1(X)) = \mathcal{F}_1(I)$, in particular, $\mathcal{NB}(\mathcal{F}_1(X))$ is an arc.

Set $I = [0, 2] \times \{0\}$ and $X = W \cup \rho_1(W) \cup I$. Notice that X is a continuum. The equality $\mathcal{NB}(\mathcal{F}_1(X)) = \mathcal{F}_1(I)$ follows from Proposition 4.7 and the fact that I is co-quasi-terminal continuum-wise connected closed subset of X satisfying that $D(X, I) = 2$ and $\mathcal{F}_1(I) \subseteq \mathcal{NWC}(X)$.

Example 4.12. There exists a continuum X such that $\mathcal{NB}(\mathcal{F}_1(X))$ is an order arc, in particular, $\mathcal{NB}(\mathcal{F}_1(X))$ is an arc.

In order to define X, set $Z = Q \cup \varphi_2 \circ \lambda_{-1}(W)$. Observe that Z is a continuum. Let X be the quotient space $Z/\{\overline{2}, (0, 1)\}$ obtained by shrinking the closed subset $\{\overline{2}, (0, 1)\}$ of Z to a point. We have that X is a continuum. If there is no risk of confusion, each point of X will be denoted as a point in Z. Set $I = \{0\}\times[0, 1]$. Let us prove that $\mathcal{NB}(\mathcal{F}_1(X))$ is an arc by showing that $\mathcal{NB}(\mathcal{F}_1(X)) = \{\{0\} \times [0, r] : r \in [0, 1]\}$.

First, since I is co-quasi-terminal continuum-wise connected subset of X, by Proposition 4.5, $\mathcal{NB}(\mathcal{F}_1(X))$ is contained in $\langle I\rangle$. Now, observe that I has outlet end points. In light of Proposition 4.8, we have that $\mathcal{NB}(\mathcal{F}_1(X)) \subseteq \mathcal{C}(I)$. On the other hand, each $Z \in \mathcal{C}(X)$ satisfying that $(0, 0) \in Z$ and $Z \setminus I \neq \emptyset$ must contain I. Hence, if $A \in \mathcal{C}(I)$ and $(0, 0) \in X \setminus A$, then $\kappa((0, 0), X \setminus A)$ is contained in I and so, by (b) of Theorem 2.4, $A \notin \mathcal{NB}(F_1(X))$. In conclusion, $\mathcal{NB}(\mathcal{F}_1(X)) \subseteq \{A \in \mathcal{C}(I) : (0, 0) \in A\} = \{\{0\} \times [0, r] : r \in [0, 1]\}$.

Finally, observe that if $r \in [0, 1]$, then $\{0\} \times [0, r] \in \mathcal{NWC}(X)$. This condition and Theorem 3.1 together show that $\{\{0\} \times [0, r] : r \in [0, 1]\} \subseteq \mathcal{NB}(\mathcal{F}_1(X))$.

In according to Example 4.11, Example 4.12 and [5, Theorem 4.4 and Corollary 5.4, pp. 3617, 3618], we present the following questions.

Question 4.13. *Are the arc and the simple closed curve the unique1-dimensional continua Y such that $\mathcal{NB}(\mathcal{F}_1(X))$ is homeomorphic to Y, for some continuum X?*

Question 4.14. *For which finite graph Y there exists a continuum X such that $\mathcal{NB}(\mathcal{F}_1(X))$ is homeomorphic to Y?*

For $m \geq 3$, a *simple m-od* is a continuum homeomorphic to the cone over a discrete space with m elements.

Question 4.15. *For which $m \geq 3$ there exists a continuum X such that $\mathcal{NB}(\mathcal{F}_1(X))$ is homeomorphic to a simple m-od?*

Theorem 4.16. *For each compact metric space Y, there exists a continuum X such that $\mathcal{NB}(\mathcal{F}_1(X))$ is homeomorphic to 2^Y.*

Proof. Let Y be a compact metric space. Denote by C the Cantor set. By [13, Theorem 7.7, p. 106], there exists an onto mapping $f : C \to Y$. Let $\mathfrak{D}$ be the usc decomposition of $W \times C$ given by

$$\mathfrak{D} = \{\{\overline{1}\} \times C\} \cup \{\{\overline{0}\} \times f^{-1}(y) : y \in Y\} \cup \{\{(x,y)\} : (x,y) \in (W \setminus \{\overline{0}, \overline{1}\}) \times C\}.$$

Denote by X the decomposition space and by $q : W \times C \to X$ the natural quotient mapping. From [13, Theorem 3.9, p. 40], it follows that X is a compact metric space. Also, X is connected and so the space X is a continuum. The subspace $q(\{0\} \times C)$ of X is homeomorphic to Y (see [13, Theorem 3.21, p. 44]). Let us argue that $\mathcal{NB}(\mathcal{F}_1(X)) = \langle q(\{0\} \times C)\rangle$.

First, notice that $q(\{0\} \times C)$ is a co-quasi-terminal continuum-wise connected closed subset of X. Then, by Proposition 4.5, the inclusion $\mathcal{NB}(\mathcal{F}_1(X)) \subseteq \langle q(\{0\} \times C)\rangle$ holds.

Now, for each $A \in \langle q(\{0\} \times C)\rangle$, the subset $X \setminus A$ is continuum-wise connected. Then $\langle q(\{0\} \times C)\rangle \subseteq \mathcal{NWC}(X)$. This condition and Theorem 3.1 together prove that $\langle q(\{0\} \times C)\rangle \subseteq \mathcal{NB}(\mathcal{F}_1(X))$.

Finally, since $\mathcal{NB}(\mathcal{F}_1(X)) = \langle q(\{\overline{0}\} \times C)\rangle$ and $q(\{\overline{0}\} \times C)$ is homeomorphic to Y, we obtain that $\mathcal{NB}(\mathcal{F}_1(X))$ is homeomorphic to 2^Y. □

The next example gives a partial answer to [6, Question 4.8, p. 102] by taking a continuum Y and applying [9, Corollary 14.10, p. 114].

Proposition 4.17. *If Y is either a Cantor set or a Hilbert cube, then there exists a continuum X such that $\mathcal{NB}(\mathcal{F}_1(X))$ is homoemorphic to Y.*

Proof. The hyperspace 2^Y is Cantor set when Y is Cantor set. Thus, by Theorem 4.16, there exists a continuum X such that $\mathcal{NB}(\mathcal{F}_1(X))$ is homeomorphic to the Cantor set.

Now, the hyperspace 2^Y is homeomorphic to a Hilbert cube provided that Y is a Peano continuum (see [9, Theorem 11.3, p. 89]). Then, applying Theorem 4.16, we get that there exists a continuum X such that $\mathcal{NB}(\mathcal{F}_1(X))$ is homoemorphic to a Hilbert cube □

Naturally, the following question arises.

Question 4.18. *For which continuum Y there exists a continuum X such that $\mathcal{NB}(\mathcal{F}_1(X))$ is homeomorphic to $\mathcal{C}(Y)$?*

Particularly, an arc is an example for the last question.

Example 4.19. There exists a continuum X containing an arc I such that $\mathcal{NB}(\mathcal{F}_1(X)) = \mathcal{C}(I)$, in particular, $\mathcal{NB}(\mathcal{F}_1(X))$ is a 2-cell.

Set $Y = Q \cup \rho_1(W) \cup \varphi_2 \circ \lambda_{-1}(W)$. Let $\mathfrak{D}$ be the decomposition of Y whose non-degenerate elements are the closed subsets $\{(0,0),(0,2)\}$ and $\{(0,1),\overline{2}\}$. Notice that $\mathfrak{D}$ is an upper semicontinuous decomposition. So, the decomposition space, denoted by X, is a continuum (see [13, Theorem 3.10, p. 40]). If there is no risk of confusion, each point of X will be represented as a point of Y. Set $I = \{0\} \times [0,1]$. Let us prove that $\mathcal{NB}(\mathcal{F}_1(X)) = \mathcal{C}(I)$.

First, notice that I is co-quasi-terminal continuum-wise connected. Invoke Proposition 4.5 to prove that $\mathcal{NB}(\mathcal{F}_1(X)) \subseteq \langle I\rangle$. On the other hand, I has outlet end points. Then, by Proposition 4.8, $\mathcal{NB}(\mathcal{F}_1(X)) \subseteq \mathcal{C}(I)$.

Now, for each $A \in \mathcal{C}(I)$, the subset $X \setminus A$ of X is continuum-wise connected. Then $\mathcal{C}(I) \subseteq \mathcal{NWC}(X)$. By Theorem 3.1, we have that $\mathcal{C}(I) \subseteq \mathcal{NB}(\mathcal{F}_1(X))$.

In conclusion, $\mathcal{NB}(\mathcal{F}_1(X)) = \mathcal{C}(I)$. From [9, Example 5.1, p. 33], it follows that $\mathcal{NB}(\mathcal{F}_1(X))$ is homeomorphic to a 2-cell.

Example 4.20. There exists a family $\mathcal{E}$ of continua such that $\{\mathcal{NB}(\mathcal{F}_1(X)) : X \in \mathcal{E}\}$ is an uncountable incomparable family of continua (*incomparable family* means that there is no an onto mapping between two different members of the family).

Let Y be a compactification of a ray $[0,\infty)$ with an arc as remainder. In light of [12, Lemma 11, p. 131], we may assume that $Y = I \cup R \subseteq [0,1] \times [0,1] \times \{0\}$ where $I = \{0\} \times [0,1] \times \{0\}$ and R is a topological copy of the ray $[0,\infty)$ such that $(1,1,0)$ is its unique end point. Now, set $W_1 = \{(\frac{x}{2}, 1, y) : (x,y) \in W \cup \rho_1(W)\}$, $W_2 = \{(\frac{x}{2}, \frac{x}{2}, y) : (x,y) \in W \cup \rho_1(W)\}$ and $X = Y \cup W_1 \cup W_2$. Then X is a continuum and W_1 and W_2 are topological copies of $W \cup \rho_1(W)$. We shall prove that $\mathcal{NB}(\mathcal{F}_1(X)) = \mathcal{C}(I) \cup \mathcal{F}_1(R)$.

First, observe that Y is a co-quasi-terminal continuum-wise connected subset of X. Invoke Proposition 4.5 to show that $\mathcal{NB}(\mathcal{F}_1(X)) \subseteq \langle Y \rangle$. Since $D(X,R) = 2$, Proposition 4.6 guarantees that $\mathcal{NB}(\mathcal{F}_1(X)) \cap \langle R \rangle \subseteq \mathcal{F}_1(R)$. On the other hand, since I has outlet end points, by Proposition 4.8, we have that $\mathcal{NB}(\mathcal{F}_1(X)) \cap \langle I \rangle \subseteq \mathcal{C}(I)$. Next, let us argue that $\mathcal{NB}(\mathcal{F}_1(X)) \cap \langle Y, I, R \rangle$ is empty. Let $A \in \langle Y, I, R \rangle$ be such that $\operatorname{Int} A = \emptyset$. Assume that $h : [0,\infty) \to R$ is a homeomorphism and set $t = \inf h^{-1}(A)$. Since $h((t,\infty))$ is an open subset of X, there exists $s \in (t,\infty)$ such that $h(s) \notin A$. Notice that $\kappa(h(s), X \setminus A)$ is contained in $h([t,\infty))$ and so it is not a dense subset of X. Thus, by (b) of Theorem 2.4, $A \notin \mathcal{NB}(\mathcal{F}_1(X))$. So, from the equality $\langle Y \rangle = \langle I \rangle \cup \langle R \rangle \cup \langle Y, I, R \rangle$, it follows that $\mathcal{NB}(\mathcal{F}_1(X)) \subseteq \mathcal{C}(I) \cup \mathcal{F}_1(R)$.

Now, observe that $\mathcal{C}(I) \cup \mathcal{F}_1(R) \subseteq \mathcal{NWC}(X)$. By Theorem 3.1, $\mathcal{NB}(\mathcal{F}_1(X))$ contains $\mathcal{C}(I) \cup \mathcal{F}_1(R)$.

Finally, in light of [1, Theorem 4.7, p. 244], there exists an uncountable family $\mathcal{E}$ of incomparable of compactifications of the ray $[0,\infty)$ with an arc as remainder. Then $\{\mathcal{NB}(\mathcal{F}_1(X)) : X \in \mathcal{E}\}$ is uncountable incomparable family of continua.

Finally, we pose the following question.

Question 4.21. *For which continuum Y there exists a continuum X such that $\mathcal{NB}(\mathcal{F}_1(X))$ is homeomorphic to Y?*

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